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Small Class Numbers and Extreme Values of L-Functions of Quadratic Fields

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Abstract. The table of class numbers h of imaginary quadratic fields described in [1] was placed on magnetic tape. This tape was then processed to find the occurrences of $h \leqslant 125$ and to find the successive extreme values of the Dirichlet L-functions $L(1, \chi_{-D})$, χ_{-D} the Kronecker symbol of the field $Q(\sqrt{-D})$ of discriminant $-D$. A comparison was made between the observed extrema and the bounds obtained for the L-functions by Littlewood [5] assuming Riemann hypotheses.

Introduction. Recently, a computation was made of the class numbers and class groups of imaginary quadratic fields of discriminant $-D$, for $0 < D < 4000000$. In [1], we discussed that computation and summarized some of the results. Since then, the data have been put on magnetic tape for permanent storage and ready access. At the suggestion of Daniel Shanks, we undertook to reexamine the data with a view to confirming and extending the results in a review of Lakein and Kuroda [2], in Shanks [3] and [8], and in Lehmer, Lehmer, and Shanks [4]. In [2], the first and probably last appearances of odd class numbers $h \leqslant 49$ are listed. In Section 1 of this paper, we confirm this list with our own tables and extend it to all $h \leqslant 125$. The list seems to be complete. We also list, in the tables at the end, the first occurrences of specific class groups of these small orders. The rest of our paper concerns the successive extreme values of the L-functions of real characters of these imaginary quadratic fields and extends some of the tables of [3] and [4]. These two topics are connected: A maximum of the L-function can only occur at the first appearance of a given class number; and, unless minima occur for different discriminants having the same class number, the minimum can only occur at the last appearance of a given class number.

All of the computations were done on the IBM 370/158 computer at the University of Illinois at Chicago Circle, Chicago, Illinois; we thank the University for the use of the computer facilities.

1. Small Class Numbers and Class Groups. In [2], the first and last occurrences of odd class numbers $h = h(Q(\sqrt{-D}))$ less than 50 are exhibited in two tables, one for $D \equiv 7 \pmod 8$, and one for $D \equiv 3 \pmod 8$, for $0 < D < 465071$, D prime. In Tables 1, 2, and 3, we present, for $0 < D < 4000000$ (D no longer necessarily prime), the first and last occurrences of all class numbers $h \leqslant 125$, for even D as well as the two congruence classes modulo 8. We also include the number of examples of each

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TABLE 1

Small class numbers for even discriminants

Class number H, first D, last D, number of D, with H(− D) = H

H	FIRST D	LAST D	NO.	H	FIRST D	LAST D	NO.
1	4	8	2	2	20	232	7
4	56	1012	24	6	104	1588	16
8	164	3448	57	10	296	5272	26
12	356	8248	81	14	404	9172	21
16	584	14008	138	18	1172	21652	32
20	776	21508	133	22	1076	26548	30
24	1316	32008	227	26	1256	45208	49
28	1364	53188	161	30	1844	57688	51
32	1784	65668	300	34	2456	73588	45
36	2504	85012	234	38	4916	121972	51
40	2756	120712	381	42	3176	117748	88
44	3416	137272	240	46	3764	189352	58
48	4424	162628	554	50	4436	174868	61
52	5924	201268	255	54	6296	194968	67
56	4616	249208	520	58	5144	188248	53
60	5444	332872	439	62	10484	341608	75
64	6536	424708	711	66	9236	370792	98
68	7124	411832	343	70	7796	377512	111
72	7556	539092	758	74	12776	332308	81
76	9176	486088	345	78	8564	369832	106
80	10856	435268	971	82	11156	604948	85
84	10436	667192	539	86	11864	548788	74
88	12536	601528	764	90	14804	596212	146
92	13604	566008	383	94	13844	737752	92
96	16376	795832	1429	98	15896	645208	106
100	13796	802888	544	102	22376	900568	124
104	14024	1154008	848	106	19736	997588	105
108	16964	850312	629	110	23336	1021672	119
112	17156	1011112	1333	114	21656	861268	137
116	20324	1104772	538	118	21416	914068	111
120	19304	1391812	1535	122	26456	938452	101
124	26120	1411012	521				

class number in this range of discriminants. Up to $h = 49$, our tables coincide completely with those of [2].

Although mere computation cannot supply an answer, one can ask whether these tables are complete—are the last examples observed actually the last examples? To assume $h \leqslant 125$ for $D \geqslant 4000000$ would imply that $L(1, \chi_{-D}) < 125\pi/2000 = .1963$. For $D \equiv 7 \pmod 8$, the observed minimum of the L-function for $D < 4000000$ (see Section 2) is three times that. To find such a large D of this congruence class with so small a class number would be quite surprising, but the nonexistence of such D has only been proved for $h \leqslant 13$ [8].

For $D \equiv 3 \pmod 8$, the minimum of the L-function is .1988, which occurs for the very unusual discriminant -991027, of class number 63 (see [3], [7]). It does not seem completely unlikely, therefore, for other examples with $h \leqslant 125$ to occur. However, as a glance at Table 3 shows, it is likely that few more, if any, will exist; the largest D in this table is smaller than 3.1 million.

For the even discriminants, it is much more likely that this list is complete. The minimum of the L-function, .3041, occurs as early as $D = 1154008$, and all entries in Table 1 are smaller than 1.5 million.

TABLE 2

Small class numbers for discriminants 1 mod 8

Class number H, first D, last D, number of D, with H(− D) = H

H	FIRST D	LAST D	NO.	H	FIRST D	LAST D	NO.
1	7	7	1	2	15	15	1
3	23	31	2	4	39	55	2
5	47	127	4	6	87	247	2
7	71	487	5	8	95	583	5
9	199	1423	5	10	119	415	6
11	167	1303	4	12	231	1807	8
13	191	2143	6	14	215	2335	10
15	239	2647	6	16	399	4687	17
17	383	4447	5	18	335	3543	12
19	311	5527	11	20	455	4855	18
21	431	5647	7	22	591	8143	21
23	647	6703	10	24	695	9823	13
25	479	5503	6	26	551	10543	14
27	983	11383	12	28	831	9367	29
29	887	8863	11	30	671	13423	17
31	719	13687	10	32	791	10327	28
33	839	13183	19	34	1079	12247	21
35	1031	12007	8	36	959	18943	39
37	1487	22807	11	38	1199	19063	19
39	1439	18127	16	40	1271	17647	44
41	1151	21487	16	42	1959	18247	21
43	1847	22303	16	44	1391	25327	56
45	1319	29863	15	46	2615	30823	23
47	3023	25303	10	48	1751	30487	62
49	1511	27127	12	50	1799	32167	37
51	1559	28087	18	52	1679	32767	62
53	2711	39103	16	54	2759	43327	35
55	4463	29383	14	56	1991	36103	62
57	2591	44647	21	58	2231	41407	30
59	2399	46447	16	60	2159	56743	70
61	3863	53887	17	62	2471	49303	25
63	2351	43543	17	64	2519	54223	64
65	3527	46687	28	66	3431	74743	52
67	3719	65167	24	68	2831	61807	72
69	3119	68863	23	70	3239	64807	47
71	5471	65983	26	72	3311	66487	85
73	2999	73327	18	74	4151	59767	32
75	4703	77863	27	76	3071	87607	79
77	6263	71983	20	78	5111	100783	48
79	4391	90127	25	80	5183	109783	98
81	3671	63823	28	82	3839	96583	36
83	3911	118903	24	84	4031	107503	96
85	4079	97927	25	86	6767	112183	41
87	5279	105967	19	88	4199	121927	113
89	6311	98407	14	90	5951	121207	76
91	4679	122263	35	92	4991	120367	103
93	5351	86263	27	94	7367	153847	50
95	6959	150847	32	96	6071	176863	157
97	5519	115663	23	98	6191	129343	40
99	5591	126823	25	100	7991	137503	108
101	5879	170503	24	102	9383	152527	60
103	13799	164767	20	104	11535	161767	127
105	6719	151783	48	106	7631	142927	56
107	8231	148927	30	108	5759	213007	135
109	5711	120103	20	110	7751	180343	59
111	15359	206383	36	112	7895	199543	157
113	8039	176023	27	114	6431	195463	64
115	7559	231823	29	116	7271	216847	113
117	6551	181927	40	118	8351	256183	59
119	9791	263167	25	120	7391	229783	180
121	8111	202183	32	122	11735	149167	59
123	14087	220903	33	124	9119	227287	116
125	11519	192343	23				

TABLE 3

Small class numbers for discriminants 5 mod 8

Class number H, first D, last D, number of D, with H(− D) = H

H	FIRST D	LAST D	NO.	H	FIRST D	LAST D	NO.
1	3	163	6	2	35	427	10
3	59	907	14	4	155	1555	28
5	131	2683	21	6	339	3763	33
7	251	5923	26	8	299	6307	69
9	419	10627	29	10	611	13843	55
11	659	15667	37	12	731	17803	117
13	1019	20563	31	14	899	30067	64
15	971	34483	62	16	1139	31243	167
17	1091	37123	40	18	1691	48427	106
19	2099	38707	36	20	1739	58507	199
21	1931	61483	78	22	2435	85507	88
23	1811	90787	58	24	2219	111763	271
25	3851	93307	89	26	2651	103027	127
27	3299	103387	81	28	3419	126043	267
29	2939	166147	72	30	4835	134467	187
31	3251	133387	63	32	5795	164803	380
33	4091	222643	82	34	4331	189883	153
35	4259	210907	95	36	4619	217627	395
37	8147	158923	74	38	7091	289963	167
39	5099	253507	99	40	7739	260947	487
41	9467	296587	93	42	6179	280267	230
43	6299	300787	90	44	5771	319867	395
45	6971	308323	139	46	7619	462883	187
47	8291	375523	97	48	9779	335203	749
49	8819	393187	120	50	8531	389467	247
51	14771	546067	141	52	11339	439147	453
53	22619	425107	98	54	11219	532123	325
55	9539	452083	149	56	11891	494323	623
57	13331	615883	158	58	13571	586987	208
59	18443	474307	112	60	12971	662803	793
61	11171	606643	115	62	11051	647707	223
63	16979	991027	199	64	14531	693067	897
65	12011	703123	136	66	16715	958483	380
67	13859	652723	96	68	16379	819163	561
69	16931	888427	186	70	14339	811507	402
71	17939	909547	124	72	15539	947923	1087
73	28211	886867	101	74	18851	951043	294
75	19211	916507	210	76	24155	1086187	651
77	24251	1242763	196	78	20891	1004347	407
79	20411	1333963	150	80	21251	1165483	1208
81	19379	1030723	200	82	20459	1446547	281
83	35531	1074907	126	84	22931	1225387	1080
85	23099	1285747	196	86	24779	1534723	357
87	22571	1261747	203	88	25091	1265587	1028
89	28619	1429387	178	90	25451	1548523	579
91	26171	1391083	179	92	31811	1452067	762
93	28859	1475203	235	94	31499	1587763	367
95	27011	1659067	209	96	28811	1684027	1697
97	37619	1842523	162	98	26771	2383747	434
99	28019	1480627	264	100	34859	1856563	1084
101	38891	1701043	172	102	33659	1765003	480
103	31379	1790443	161	104	31979	1853923	1165
105	37691	2317723	331	106	46571	1924267	368
107	48539	2404147	159	108	33491	2127283	1273
109	39779	1945483	189	110	46331	2100547	536
111	46619	1990123	278	112	48971	2804587	1704
113	45491	2341747	212	114	36539	2143843	542
115	42899	2458003	278	116	38099	2888923	1001
117	41411	2341243	305	118	38939	3060787	410
119	41051	2463883	251	120	43931	3086323	2198
121	44651	2145067	209	122	48851	2608747	434
123	59699	2849683	266	124	51611	2696467	1023
125	48611	2944363	267				

TABLE 4

First occurrence of small noncyclic 2-groups

Group	Even D	D≡7	D≡3
4x4	6052	***	2379
4x8	6392	***	5795
2x4x4	6360	***	43435
4x12	10808	8103	27995
4x16	7544	4895	21395
8x8	25988	***	32331
2x4x8	33288	***	50235
2x2x4x4	42420	***	105315
4x20	28024	12207	9503
4x24	24056	13359	70131
2x4x12	43160	***	86691
4x28	29444	13727	37407

TABLE 5

First occurrence of small noncyclic 3-groups

Group	Even D	D≡7	D≡3
3x3	***	***	4027
3x6	9748	***	12067
3x9	***	***	3299
3x12	3896	6583	12131
6x6	15544	***	18555
3x15	***	25447	42859
3x18	38296	5703	11651
3x21	***	***	117043
3x24	21668	8751	51995
6x12	20276	***	39819
2x6x6	26760	***	40299
3x27	***	***	45131
9x9	***	***	134059
3x30	34088	36807	91643
3x33	***	65407	54251
3x36	27656	62527	64571
6x18	57336	31983	124395
3x39	***	28279	99707

TABLE 6

First occurrence of small noncyclic 5- and 7-groups

Group	Even D	D≡7	D≡3
5x5	***	***	12451
5x10	17944	***	33531
5x15	***	38047	112643
5x20	115812	11199	173243
10x10	58424	***	63411
5x25	***	258563	320659
7x7	***	***	63499
7x14	159592	***	124043

TABLE 7

Successive minima of $L(1)$ *for even discriminants*
Negative discriminant D, $D/4$, $L(1)$, *and* LLI_D

D	D/4	L(1)	LLI_D
4	1	0.7854	0.3704
88	22	0.6698	1.4495
148	37	0.5165	1.1996
232	58	0.4125	1.0094
1012	253	0.3950	1.1031
21652	5413	0.3843	1.2765
45208	11302	0.3842	1.3155
53188	13297	0.3814	1.3144
105172	26293	0.3681	1.3009
120712	30178	0.3617	1.2844
121972	30493	0.3418	1.2143
189352	47338	0.3321	1.1974
332872	83218	0.3267	1.1994
424708	106177	0.3085	1.1410
539092	134773	0.3081	1.1475
1154008	288502	0.3041	1.1575

TABLE 8

Successive minima of $L(1)$ *for discriminants* 1 mod 8
Negative discriminant D, $L(1)$, $L_D(1)$, *and* LLI_D

D	L(1)	$L_D(1)$	LLI_D
7	1.1874	0.5937	1.0317
463	1.0220	0.5110	1.4888
487	0.9965	0.4983	1.4565
823	0.9856	0.4928	1.4882
1087	0.8576	0.4288	1.3158
1423	0.7495	0.3748	1.1671
4687	0.7342	0.3671	1.2117
176863	0.7171	0.3586	1.3461
577663	0.7110	0.3555	1.3777
669463	0.6950	0.3475	1.3518
678463	0.6903	0.3452	1.3432
773767	0.6607	0.3304	1.2898
1543063	0.6449	0.3225	1.2800
2185423	0.6375	0.3188	1.2755
2430943	0.6287	0.3143	1.2607
3668527	0.6233	0.3116	1.2613
3855223	0.6096	0.3048	1.2350

In Tables 4, 5, and 6 we list the first occurrences of noncyclic class groups of orders $\leqslant 125$. In these, the notation $N \times M$ signifies a group $C(N) \times C(M)$, with $C(N)$ a cyclic group of order N. The symbol *** means that no example occurred. Also, as in [1], the term "noncyclic 2-group" is used for a group whose principal genus is noncyclic. It was proved in [8] that, for $D \equiv 7 \pmod 8$, the groups 3×3, 5×5, 7×7 and 11×11 do not occur. If our list of small class numbers is complete, then neither 11×11 nor $5 \times 5 \times 5$ occur for imaginary quadratic fields. The groups $3 \times 3 \times 3$ and $4 \times 4 \times 4$ and $3 \times 3 \times 3 \times 3$ are also missing here and are even more likely to be nonexistent for imaginary quadratic fields since any such example would have an improbably small value of $L(1, \chi)$.

TABLE 9

Successive minima of L(1) for discriminants 5 mod 8
Negative discriminant D, L(1), LLI, $L_D(1)$, and LLI_D

D	L(1)	LLI	$L_D(1)$	LLI_D
3	0.6046	0.1231	0.9069	1.1917
43	0.4791	1.3744	0.7186	1.6999
67	0.3838	1.1937	0.5757	1.4305
163	0.2461	0.8675	0.3691	0.9958
85507	0.2364	1.2437	0.3545	1.3026
111763	0.2255	1.1981	0.3383	1.2531
166147	0.2235	1.2036	0.3353	1.2564
222643	0.2197	1.1946	0.3296	1.2453
462883	0.2124	1.1814	0.3186	1.2279
958483	0.2118	1.2029	0.3177	1.2468
991027	0.1988	1.1302	0.2982	1.1714

TABLE 10

Successive maxima of L(1) for even discriminants
Negative discriminant D, D/4, L(1), and ULI_D

D	D/4	L(1)	ULI_D
4	1	0.7854	1.3500
8	2	1.1107	0.8518
20	5	1.4050	0.7190
56	14	1.6793	0.6770
104	26	1.8483	0.6758
164	41	1.9625	0.6763
356	89	1.9980	0.6335
404	101	2.1882	0.6856
776	194	2.2555	0.6682
1256	314	2.3048	0.6585
1364	341	2.3818	0.6765
2756	689	2.3937	0.6494
4616	1154	2.5894	0.6817
7556	1889	2.6022	0.6673
8564	2141	2.6479	0.6747
13796	3449	2.6747	0.6661
14024	3506	2.7590	0.6865
22244	5561	2.7805	0.6777
25016	6254	2.7808	0.6744
32504	8126	2.7881	0.6687
35096	8774	2.8173	0.6736
42836	10709	2.8840	0.6841
52664	13166	2.9022	0.6829
61844	15461	2.9561	0.6913
92804	23201	3.0113	0.6938
98276	24569	3.0465	0.7005
120056	30014	3.0827	0.7038
324596	81149	3.0879	0.6824
326504	81626	3.1778	0.7021
650744	162686	3.2402	0.7013
973496	243374	3.3242	0.7113
2577896	644474	3.3459	0.6978
3357416	839354	3.3742	0.6991
3519764	879941	3.3926	0.7021

TABLE 11

Successive maxima of $L(1)$ *for discriminants* 1 mod 8

Negative discriminant D, $L(1)$, *ULI*, $L_D(1)$, *and* ULI_D

D	L(1)	ULI	$L_D(1)$	ULI_D
7	1.1874	0.5007	0.5937	0.2769
15	1.6223	0.4572	0.8112	0.3231
23	1.9652	0.4828	0.9826	0.3656
39	2.0122	0.4351	1.0061	0.3488
47	2.2912	0.4771	1.1456	0.3885
71	2.6099	0.5053	1.3049	0.4231
119	2.8799	0.5168	1.4399	0.4445
191	2.9551	0.5002	1.4776	0.4383
215	2.9996	0.5009	1.4998	0.4407
239	3.0482	0.5032	1.5241	0.4443
311	3.3847	0.5438	1.6924	0.4839
479	3.5886	0.5535	1.7943	0.4981
671	3.6384	0.5453	1.8192	0.4943
959	3.6521	0.5322	1.8261	0.4858
1151	3.7966	0.5458	1.8983	0.4998
1319	3.8926	0.5542	1.9463	0.5086
1511	3.9602	0.5585	1.9801	0.5137
1559	4.0579	0.5710	2.0289	0.5255
2351	4.0819	0.5592	2.0410	0.5177
2999	4.1878	0.5652	2.0939	0.5249
3071	4.3085	0.5806	2.1542	0.5394
5711	4.5313	0.5896	2.2656	0.5516
6551	4.5413	0.5866	2.2707	0.5496
8399	4.5935	0.5858	2.2967	0.5502
10391	4.7153	0.5951	2.3577	0.5599
13439	4.7154	0.5878	2.3577	0.5543
13991	4.7277	0.5883	2.3638	0.5549
14951	4.7532	0.5896	2.3766	0.5565
15791	4.8751	0.6032	2.4375	0.5696
18191	4.9614	0.6100	2.4807	0.5766
31391	5.1244	0.6155	2.5622	0.5841
38639	5.1942	0.6186	2.5971	0.5878
45239	5.1992	0.6153	2.5996	0.5853
63839	5.2595	0.6143	2.6298	0.5855
88919	5.3204	0.6138	2.6602	0.5862
95471	5.4193	0.6236	2.7096	0.5958
118271	5.5358	0.6323	2.7679	0.6047
201431	5.6559	0.6345	2.8279	0.6083
331679	5.7168	0·6312	2.8584	0.6065
366791	5.8149	0.6401	2.9075	0.6153
514751	5.8982	0.6426	2.9491	0.6186
628319	5.9252	0.6418	2.9626	0.6183
701399	5.9306	0.6404	2.9653	0.6171
819719	5.9856	0.6435	2.9928	0.6204
890951	6.0509	0.6490	3.0254	0.6259
1238639	6.0690	0.6451	3.0345	0.6229
1339439	6.1402	0.6513	3.0701	0.6290
2155919	6.3503	0.6652	3.1752	0.6434

2. Extreme Values of L-functions. Let $-D$ be the discriminant of an imaginary quadratic number field, hence $D \equiv 0$ or $3 \pmod 4$, and D is squarefree except for the possible factor of 4. For the real character $\chi_D(n) = (-D/n)$ = the Kronecker symbol modulo D, the L-function is

$$L(s, \chi_{-D}) = \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) n^{-s} = \prod_{p \text{ prime}} \left(\frac{p^s}{p^s - (-D/p)}\right). \tag{1}$$

TABLE 12

Successive maxima of $L(1)$ for discriminants 5 mod 8

Negative discriminant D, $L(1)$, $L_D(1)$, and ULI_D

D	L(1)	$L_D(1)$	ULI_D
3	0.6046	0.9069	0.5594
11	0.9472	1.4208	0.5994
35	1.0621	1.5931	0.5598
59	1.2270	1.8405	0.6085
131	1.3724	2.0586	0.6301
251	1.3881	2.0821	0.6047
299	1.4535	2.1802	0.6251
899	1.4669	2.2003	0.5876
971	1.5123	2.2684	0.6030
1091	1.6169	2.4254	0.6405
1811	1.6979	2.5469	0.6545
3251	1.7081	2.5621	0.6398
5099	1.7158	2.5737	0.6297
5771	1.8196	2.7294	0.6642
11051	1.8529	2.7793	0.6584
12011	1.8633	2.7949	0.6600
26771	1.8817	2.8225	0.6470
39731	1.9859	2.9788	0.6736
42059	1.9914	2.9871	0.6742
66491	2.0103	3.0154	0.6705
74051	2.0203	3.0305	0.6716
112811	2.0204	3.0305	0.6630
143051	2.0599	3.0899	0.6712
246419	2.0631	3.0947	0.6620
262499	2.0848	3.1272	0.6678
275651	2.1003	3.1504	0.6718
412331	2.1135	3.1703	0.6688
437051	2.1669	3.2504	0.6847
954971	2.2150	3.3225	0.6862
1692851	2.2697	3.4045	0.6937
3230939	2.2721	3.4082	0.6845
3724811	2.2789	3.4184	0.6844

Then, as is well known,

$$L(1) = L(1, \chi_{-D}) = 2h\pi/w\sqrt{D}, \tag{2}$$

where h is the class number of the field $Q(\sqrt{-D})$, and w is the number of roots of unity in the field. An exceptionally large or small class number, relative to $\sqrt{D}$, corresponds to a large or small value of $L(1)$. Certain restrictions on the size of $L(1)$ may exist, however. Assuming the validity of the Riemann hypothesis for the L-functions, Littlewood [5] deduced, for large values of D, the bounds

$$\frac{1}{\{1 + o(1)\}(12/\pi^2)e^\gamma \log\log D} < L(1) < \{1 + o(1)\}2e^\gamma \log\log D. \tag{3}$$

It is clear from the Euler product that, for odd D, the factor of 2 or 2/3 for the prime 2 is the most significant single factor. This accounts for the great difference in magnitudes of D in Tables 2 and 3. Since the values of $L(1)$ are, in a sense, a measure of the quadratic residue symbol distribution for $-D$, it is desirable to remove this strong dependence on the prime 2. In [3] and [4] the following approach was used: Define, for any positive integer D,

$$L_D(1) = \sum_{n=1}^{\infty} \left(\frac{-4D}{n}\right) n^{-1} = \prod_{p \text{ prime}} \left(\frac{p}{p - (-4D/p)}\right). \tag{4}$$

For squarefree D we note:

$$\begin{aligned} L_D(1) &= L(1, \chi_{-4D}), & D &\equiv 1, 2 \pmod 4, \\ &= (1/2)L(1, \chi_{-D}), & D &\equiv 7 \pmod 8, \\ &= (3/2)L(1, \chi_{-D}), & D &\equiv 3 \pmod 8. \end{aligned} \tag{5}$$

Shanks [3] then strengthens the bounds (3) for this modified L-function and gives

$$\frac{1}{\{1 + o(1)\}(8/\pi^2)e^\gamma \log\log 4D} < L_D(1) < \{1 + o(1)\}e^\gamma \log\log 4D. \tag{6}$$

This can be interpreted in terms of class numbers as follows: When we compare various $L_D(1)$, for odd $D \equiv 3 \pmod 4$, we are comparing the class numbers, not of the maximal orders of discriminant $-D$, but of the orders of discriminants $-4D$ [6, pp. 158ff.].

It is of some interest to examine how closely the Littlewood bounds are approached. For this reason, we define, for both $L(1, \chi_{-D})$ and $L_D(1)$, the *upper* and *lower Littlewood indices* as

$$\begin{aligned} ULI &= L(1)/(2e^\gamma \log\log D), \\ LLI &= L(1)(12/\pi^2)e^\gamma \log\log D, \end{aligned} \tag{7}$$

$$\begin{aligned} ULI_D &= L_D(1)/(e^\gamma \log\log 4D), \\ LLI_D &= L_D(1)(8/\pi^2)e^\gamma \log\log 4D. \end{aligned} \tag{8}$$

(Remark on notation: Our (7) is Shanks' (3), page 267, and our (8) is his (13), page 270 [3].)

If the Riemann hypothesis holds, then the bounds (2) and (6) become inequalities

$$ULI < 1, \quad LLI > 1, \quad ULI_D < 1, \quad LLI_D > 1, \tag{9}$$

with some allowance, of course, for the $o(1)$. We have computed the successive minima and maxima of $L(1)$ for the three types of fundamental discriminants within the range of our tables. These extreme values, with the appropriate Littlewood indices, are listed in Tables 7–12. Not surprisingly, we notice that only three examples, $D = 3$, 4, and 163, violate (9). Shanks has shown [3] that for $D = 163$ the $o(1)$ is probably large enough to accommodate the Littlewood bounds. One would expect the other two cases to be even less suspect because the bounds are, after all, asymptotic.

Thus, our computation gives no reason to believe that the Riemann hypothesis is false for these L-functions.

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